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RELATIONS BETWEEN THE FIRST INTEGRALS OF A NONHOLONOMIC MECHANICAL SYSTEM AND OF THE CORRESPONDING SYSTEM FREED OF CONSTRAINTS

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We derive the necessary and sufficient conditions for obtaining the first integral of a nonholonomic system with linear homogeneous constraints from the first integral of the corresponding system freed of constraints. We present examples.

1. We consider a nonholonomic scleronomous mechanical system with the generalized coordinates $q^1, q^2, \ldots, q^n$, the doubled kinetic energy $2T = g_{\lambda\mu} q^{\cdot\lambda} q^{\cdot\mu}$ and the force function $U = U(q^\varkappa)$. The system is subject to the $n-k$ linear homogeneous constraints $\omega^p{}_\varkappa q^{\cdot\varkappa} = 0$. In what follows the Greek indices $\lambda, \mu, \nu, \ldots, \sigma$ take the values $1, 2, \ldots, n$, while the Latin ones a, b, c, d take the values $1, 2, \ldots, k$ and p, q, r take $k+1, \ldots, n$. By introducing the new variables

$$q^{\cdot\varkappa} = \alpha_a{}^\varkappa s^{\cdot a} \tag{1.1}$$

we write the equations of motion in the following form [1]:

$$Ds^{\cdot d} / dt = F^d, \qquad Ds^{\cdot d} = ds^{\cdot d} + \Gamma^d_{bc} ds^b s^{\cdot c}$$
$$F^d = G^{da} F_a = G^{da} \alpha_a{}^\varkappa Q_\varkappa = G^{da} \alpha_a{}^\varkappa \partial u / \partial q\varkappa$$
$$\Gamma^d_{cb} = G^{da} \Gamma_{a, cb}$$
$$\Gamma_{a, cb} = \Gamma_{\varkappa, \nu, \mu} \alpha_a{}^\varkappa \alpha_b{}^\mu \alpha_c{}^\nu + g_{\lambda\varkappa} \alpha_a{}^\varkappa \partial \alpha_b{}^\lambda / \partial q^\sigma \alpha_c{}^\sigma$$

The vectors α_a $(\alpha_a{}^\varkappa)$ are called the admissible vectors of the system and satisfy the condition

$$\omega_\varkappa{}^p \alpha_a{}^\varkappa = 0 \tag{1.2}$$

The matrix G^{ad} is the inverse of the matrix $G_{ab} = g_{\lambda\mu} \alpha_a{}^\lambda \alpha_b{}^\mu$. By $\Gamma_{\varkappa, \mu\nu}$ we denote the Christoffel symbols of the first kind, defined by the metric tensor $g_{\lambda\mu}$.

We consider the case when the system moves by inertia, i.e., $U = \text{const}$. As was shown in [2], in order for $\lambda_a s^{\cdot a} = c$ to be a linear integral of a nonholonomic system, it is necessary and sufficient to fulfill the following conditions:

$$\nabla_c\lambda_a + \nabla_a\lambda_c = 0 \tag{1.3}$$

Suppose that the linear integral of a nonholonomic system has been obtained from $\xi_\varkappa \dot{q}^\varkappa$ after the substitution (1.1), therefore $\lambda_a = \xi_\varkappa \alpha_a{}^\varkappa$. The vector $\xi_\varkappa$ can be obtained in an infinity of ways. Indeed, consider the two systems of vectors

$$\omega^1, \omega^2, \ldots, \omega^k, \omega^{k+1}, \ldots, \omega^n; \qquad \omega_\varkappa{}^a = g_{\varkappa\nu}\alpha_a{}^\nu$$

$$\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n; \qquad \alpha_p{}^\varkappa = g^{\varkappa\nu}\omega_\nu{}^p$$

Here the ω^p are vectors obtained from the nonholonomic constraints, the α_a are admissible vectors, the matrix $g^{\lambda\mu}$ is the inverse of matrix $g_{\lambda\mu}$. From condition (1.2) we find that $\alpha_p{}^\varkappa\omega_\varkappa{}^a = 0$. Let $\eta_\varkappa = \xi_\varkappa + \rho_p\omega_\varkappa{}^p$, where ρ_p are arbitrary functions of q^λ. Obviously, $\xi_\varkappa\alpha_a{}^\varkappa = \eta_\varkappa\alpha_a{}^\varkappa$. Substituting $\lambda_a = \xi_\varkappa\alpha_a{}^\varkappa$ into condition (1.3), we obtain

$$\alpha_c{}^\nu\alpha_a{}^\varkappa(\nabla_\nu\xi_\varkappa + \nabla_\varkappa\xi_\nu) + \xi_\varkappa(\nabla_c\alpha_a{}^\varkappa + \nabla_a\alpha_c{}^\varkappa) = 0 \tag{1.4}$$

$$\nabla_c\alpha_a{}^\varkappa = \frac{\partial\alpha_a{}^\varkappa}{\partial q_\nu}\alpha_c{}^\nu + \Gamma^\varkappa_{\lambda\mu}\alpha_c{}^\lambda\alpha_a{}^\mu - \Gamma^b_{ca}\alpha_b{}^\varkappa$$

$$\Gamma^\varkappa_{\lambda\mu} = g^{\varkappa\rho}\Gamma_{\rho,\lambda\mu}$$

where $\Gamma^\varkappa_{\lambda\mu}$ are Christoffel symbols of the second kind.

If $\xi_\varkappa\dot{q}^\varkappa = c$ is a linear integral of a constraint-freed system (we subsequently write CFS for brevity), it is known [3] that

$$\nabla_\nu\xi_\varkappa + \nabla_\varkappa\xi_\nu = 0 \tag{1.5}$$

Taking (1.5) into account, from condition (1.4) we find

$$\xi_\varkappa(\nabla_c\alpha_a{}^\varkappa + \nabla_a\alpha_c{}^\varkappa) = 0 \tag{1.6}$$

Conditions (1.6) are necessary and sufficient for that the linear integral $\lambda_a\dot{s}^a = c$ of a nonholonomic system can be obtained from the linear integral $\xi_\varkappa\dot{q}^\varkappa = c$ of the corresponding CFS. We assume that the integral $\xi_\varkappa\dot{q}^\varkappa = c$ generates the integral $\lambda_a\dot{s}^a = c$.

From the way in which the objects $\Gamma_{a,bc}$ were defined it follows that

$$g_{\lambda\mu}\alpha_b{}^\lambda\nabla_c\alpha_a{}^\mu = 0 \tag{1.7}$$

Relation (1.7) can be written as

$$\nabla_c\alpha_a{}^\varkappa = B^p_{ca}\alpha_p{}^\varkappa \tag{1.8}$$

where the $B_{ca}{}^p$ are functions of q^λ. Taking (1.8) into account, we obtain conditions (1.6) in the form

$$\xi_\varkappa\alpha_p{}^\varkappa(B^p_{ca} + B^p_{ac}) = 0 \tag{1.9}$$

The condition

$$\xi_\varkappa\alpha_p{}^\varkappa = 0 \tag{1.10}$$

is only sufficient. It is not necessary, notwithstanding the assertions in [4]. As a matter of fact, if we treat $\xi_\varkappa\alpha_p{}^\varkappa$ as being unknown, (1.9) is a linear homogeneous system of not more than $^1/_2\, k\,(k+1)$ equations in $n-k$ unknowns. Evidently when n is large, the number of unknowns is greater than the number of equations. If the mechanical system satisfies the requirement

$$B^p_{ca} + B^p_{ac} = 0 \tag{1.11}$$

condition (1.9) is satisfied identically. Consequently, every linear integral of the CFS generates a linear integral of the nonholonomic system.

Example 1. We consider the following problem [5].Two wheels of radius b connected by an axle of length $2l$ roll on a plane and may rotate freely around the axle. As the generalized coordinates we select $q^1 = \varphi$, $q^2 = x$, $q^3 = y$, $q^4 = \psi$, $q^5 = \psi'$, where x, y are the coordinates of the system's center of gravity, φ is the angle between the x-axis and the O_1O_2-axis, ψ and ψ' are angles of rotation of the wheels, measured from the vertical radius. The no-slip condition of rolling yields

$$x^{\cdot} \cos\varphi + y \sin\varphi = 0$$
$$-x^{\cdot}\sin\varphi + y^{\cdot}\cos\varphi - l\varphi^{\cdot} - b\psi^{\cdot\prime} = 0$$
$$-x^{\cdot}\sin\varphi + y^{\cdot}\cos\varphi + l\varphi^{\cdot} - b\psi^{\cdot} = 0$$

By introducing the new variables

$$q^{\cdot 1} = \varphi^{\cdot}, \qquad q^{\cdot 2} = -s^{\cdot}\sin\varphi, \qquad q^{\cdot 3} = s^{\cdot}\cos\varphi$$
$$q^{\cdot 4} = \frac{l}{b}\varphi^{\cdot} + \frac{s^{\cdot}}{b}, \qquad q^{\cdot 5} = \frac{s^{\cdot}}{b} - \frac{l}{b}\varphi^{\cdot}$$

we find

$$\alpha_1\,(1,\ 0,\ 0,\ l/b,\ -l/b), \qquad \alpha_2\,(0,\ -\sin\varphi,\ \cos\varphi,\ 1/b,\ 1/b)$$

From the constraint equations we obtain

$$\omega^3\,(0,\ \cos\varphi,\ \sin\varphi,\ 0,\ 0), \quad \omega^4\,(-l,\ -\sin\varphi,\ \cos\varphi,\ 0,\ -b) \qquad \omega^5\,(l,\ -\sin\varphi,\ \cos\varphi,\ -b,\ 0)$$

Keeping in mind that

$$g_{11} = 2ml^2 = A,\ g_{22} = g_{33} = 2m + m' = B,\ g_{44} = g_{55} = I + I'b/4l^2 = C$$

where I is the wheel's moment of inertia about the center, I' is the axle's moment of inertia about the center of gravity, m is the wheel's mass and m' is the axle's mass, we find

$$\alpha_3\left(0,\ \frac{\cos\varphi}{B},\ \frac{\sin\varphi}{B},\ 0,\ 0\right)$$
$$\alpha_4\left(-\frac{l}{A},\ -\frac{\sin\varphi}{B},\ \frac{\cos\varphi}{B},\ \frac{bD}{C^2-D^2},\ -\frac{bC}{C^2-D^2}\right)$$
$$\alpha_5\left(\frac{l}{A},\ -\frac{\sin\varphi}{B},\ \frac{\cos\varphi}{B},\ -\frac{bC}{C^2-D^2},\ \frac{bD}{C^2-D^2}\right)$$

By relations (1.8) we obtain $B_{12}{}^3 = -B$ and all remaining $B_{ab}^p = 0$. Condition (1.9) yields $\xi_\varkappa \alpha_3{}^\varkappa = 0$ or $\xi^\varkappa \omega_\varkappa{}^3 = 0$. In developed form we have $\xi_2\cos\varphi + \xi_3\sin\varphi = 0$ or $\xi^2\cos\varphi + \xi^3\sin\varphi = 0$.

If the linear integral of the corresponding CFS satisfies the condition described, it generates a linear integral of the nonholonomic system. Taking the form of $g_{\lambda\mu}$ into account, we conclude that all the coordinates of the CFS are cyclic. Hence it follows that the CFS's integrals

$$\partial T/\partial\varphi^{\cdot} = C_1, \qquad \partial T/\partial\psi^{\cdot} = C_2, \qquad \partial T/\partial\psi^{\cdot\prime} = C_3$$

generate the nonholonomic system's integrals

$$\psi = C_1,\ \frac{C-D}{b}\,l\varphi^{\cdot} + \frac{C+D}{b}\,s^{\cdot} = C_2,\ \ \frac{D-C}{b}\,l\varphi^{\cdot} + \frac{C+D}{b}\,s^{\cdot} = C_3$$

We can immediately verify that these integrals of the nonholonomic mechanical system do not satisfy condition (1.10). Condition (1.10) is geometric, i.e., it does not alter under the change of coordinate $q^{\varkappa'} = q^{\varkappa'}(q^\varkappa)$ and under the transformation of vectors $\alpha_{p'}^\varkappa = \gamma_{p'}^p\,\alpha_p^\varkappa$, when $\det \|\gamma_{p'}^p\| \neq 0$. Indeed,

$$\xi_{\varkappa'}\alpha_{p'}^{\varkappa'} = \xi_\varkappa A_{\varkappa'}^\varkappa A_\nu^{\varkappa'}\alpha_p^\nu\gamma_{p'}^p = \xi_\varkappa\alpha_p^\varkappa\gamma_{p'}^p = 0$$

The definitions of the notion of a cyclic coordinate in [4, 6] differ. Even if we follow the definition in [4], the linear integrals indicated do not correspond to cyclic coordinates. According to this definition we must have $\partial T / \partial q^{\varkappa_0} = 0$ and $\omega^p_{\varkappa_0} = 0$ [4] in Routh's equation for $q^{\varkappa_0}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{\varkappa_0}}\right) - \frac{\partial T}{\partial q^{\varkappa_0}} = \lambda_p \omega^p_{\varkappa_0}$$

It is easily verified that the integrals satisfy the first condition. Therefore, $q^{\varkappa_0}$ is a cyclic coordinate; in other words, $\xi^\varkappa_{\varkappa_0} = \delta^\varkappa_{\varkappa_0}$, where $\delta^\varkappa_{\varkappa_0}$ is the Kronecker symbol. Since $\xi^\varkappa_{\varkappa_0}\omega^p_\varkappa = \delta^\varkappa_{\varkappa_0}\omega^p_\varkappa = \omega^p_{\varkappa_0} = 0$, the second condition is not satisfied. Consequently, the linear first integrals, homogeneous in the velocities, of the nonholonomic system do not correspond to the cyclic coordinates in [4].

Example 2. Consider the free motion of a homogeneous ball of mass $m = 1$, radius a, radius of gyration k on a horizontal rough plane [6]. The motion is defined in the coordinates $q^1 = \varphi$, $q^2 = \psi$, $q^3 = \theta$, $q^4 = x$, $q^5 = y$. The doubled kinetic energy is

$$2T = \dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)$$

The nonholonomic constraints are written as

$$\dot{x} = a\sin\psi\dot{\theta} - a\sin\theta\cos\psi\dot{\varphi}$$
$$\dot{y} = -a\cos\psi\dot{\theta} - a\sin\theta\sin\psi\dot{\varphi}$$

Analogously to Example 1 we find

$$\alpha_1\,(1,\ 0,\ 0,\ -a\sin\theta\cos\psi,\ -a\sin\theta\sin\psi)$$
$$\alpha_2\,(0,\ 1,\ 0,\ 0,\ 0)$$
$$\alpha_3\,(0,\ 0,\ 1,\ a\sin\psi,\ -a\cos\psi)$$
$$\alpha_4\left(\frac{a\cos\psi}{k^2\sin\theta},\ -\frac{a\cos\theta\cos\psi}{k^2\sin\theta},\ -\frac{a\sin\psi}{k^2},1,0\right).$$
$$\alpha_5\left(\frac{a\sin\psi}{k^2\sin\theta},\ -\frac{a\cos\theta\sin\psi}{k^2\sin\theta},\ \frac{a\cos\psi}{k^2},0,1\right)$$
$$B_{21}^4 = -B_{12}^4 = B^\circ\sin\theta\sin\psi,\ B_{12}^5 = -B_{21}^5 = B^\circ\sin\theta\cos\psi$$
$$B^4_{13} = -B^4_{31} = B^\circ\cos\theta\cos\psi,\qquad B^5_{13} = -B^5_{31} = B^\circ\cos\theta\sin\psi$$
$$B^4_{32} = -B^4_{23} = B^\circ\cos\psi,\qquad B^5_{23} = -B^5_{32} = B^\circ\sin\psi$$
$$B^4_{11} = B^4_{22} = B^4_{33} = B^5_{11} = B^5_{22} = B^4_{33} = 0,\qquad B^\circ = {}^1/_2 ak^2/(a^2+k^2)$$

It follows from condition (1.11) that each linear integral of the CFS generates a linear integral of the nonholonomic system.

Upto now we have been considering the problem when the CFS's linear integral generates the nonholonomic system's linear integral. The inverse problem is stated in the following way. Suppose that we are given the integral $\lambda_a\dot{s}^a = C$ of a nonholonomic system. We need to establish whether it is generated by a linear integral of a CFS, i.e., whether we can represent λ_a in the form

$$\lambda_a = \eta_\varkappa\alpha_a{}^\varkappa \tag{1.12}$$

where $\eta_\varkappa\dot{q}^\varkappa = C$ is the CFS's linear integral. Consequently, we must verify the existence of a vector $\xi_\varkappa = \eta_\varkappa + \rho_p\omega_\varkappa{}^q$ satisfying the Killing's equation [3], and $\eta_\varkappa$ is determined from (1.12). Let us show by means of an example that not every integral of the nonholonomic system is generated by an integral of the corresponding CFS.

Example 3. Consider Chaplygin's sleigh on a horizontal plane when the direction of the runner is perpendicular to the segment connecting the center of gravity and the cutting point [7]. As was shown in [8], $x^{\cdot} / \cos\varphi = C$ is a linear integral of the nonholonomic system. In other words, $\lambda_1 = 0$, $\lambda_2 = 1/\cos\varphi$ yield a vector defining the system's linear integral. We find that $\eta_1 = 0$, $\eta_2 = \cos\varphi$, $\eta_3 = \sin\varphi$. Hence we obtain

$$\xi_1 = 0, \qquad \xi_2 = \cos\varphi - \rho\,\mathrm{tg}\,\varphi, \qquad \xi_3 = \sin\varphi + \rho$$

We can directly verify that a ρ does not exist for which $\xi_\varkappa$ satisfies the Killing equation.

In the case when $U \neq$ const (the system does not move by inertia), conditions (1.3) and

$$\lambda_a F^a = 0 \tag{1.13}$$

are the existence conditions for the nonholonomic system's linear integral of the form $\lambda_a s^{\cdot a} = C$. For the linear integral $\xi_\varkappa q^{\cdot\varkappa} = C$ of the CFS to generate a linear integral of the nonholonomic system, we must fulfill, in addition to condition (1.9), the further condition [1, 2]

$$\xi_\varkappa \alpha_a{}^\varkappa G^{ab} Q_\nu \alpha_b{}^\nu = 0 \tag{1.14}$$

Condition (1.14) is essential. With every system containing $U \neq$ const we can associate a system moving by inertia. If $\eta_\varkappa q^{\cdot\varkappa} = C$ is a CFS's linear integral, which generates a linear integral of a nonholonomic system moving by inertia, and if it also is the linear integral of a CFS which does not move by inertia, then it does not follow that this integral remains an integral of a nonholonomic system not moving by inertia.

Consider the system in Example 1. Suppose $U = U(\varphi)$. The integral $\partial T / \partial\psi^{\cdot} = C_3$ also is an integral of a CFS since $\xi^\varkappa \partial U / \partial q^\varkappa = 0$. On the other hand, condition (1.14) is not satisfied identically. Indeed,

$$2\theta = G_{11}\varphi^{\cdot 2} + G_{22}s^{\cdot 2}$$

$$G_{11} = \frac{1}{G^{11}} = A + 2\frac{l^2}{b^2}(C - D), \qquad G_{22} = \frac{1}{G^{22}} = B + \frac{2}{b^2}(C + D)$$

$$G^{12} = G^{21} = 0, \qquad Q_\varkappa \alpha_1{}^\varkappa = \partial U / \partial\varphi, \qquad Q_\varkappa \alpha_2{}^\varkappa = 0$$

The left hand side of (1.14) has the form

$$\frac{l}{b}\,\frac{D - C}{G_{11}}\,\frac{\partial U}{\partial\varphi} \neq 0$$

Consequently,

$$\frac{D - C}{b}\, l\varphi^{\cdot} + \frac{D + C}{b}\, s^{\cdot} = C_3$$

is not an integral of a nonholonomic system with $U = U(\varphi)$.

In the general case, if we bear in mind that

$$g^{\lambda\mu} = G^{ab}\alpha_a{}^\lambda \alpha_b{}^\mu + G^{pq}\alpha_p{}^\lambda \alpha_q{}^\mu \tag{1.15}$$

where the matrix G^{pq} is the inverse of matrix $G_{pq} = g_{\lambda\mu}\alpha_p{}^\lambda \alpha_q{}^\mu$, we can write condition (1.14) as

$$\xi_\varkappa Q^\varkappa - G^{pq}\alpha_p{}^\varkappa \alpha_q{}^\nu Q_\varkappa \xi_\nu = 0$$

If $\xi_\varkappa q^{\cdot\varkappa} = C$ is an integral of a CFS, $\xi_\varkappa Q^\varkappa = 0$. In order to eliminate the necessity of considering (1.14), we must fulfill the condition

$$G^{pq}\alpha_p{}^\varkappa \alpha_q{}^\nu Q_\varkappa \xi_\nu = 0 \quad \text{or} \quad G^{pq}F_p\lambda_q = 0$$

$$\lambda_q = \xi_\varkappa \alpha^{\ \varkappa}, \qquad F_p = Q_\nu \alpha_p{}^\nu \tag{1.16}$$

Relation (1.16) shows that the perpendicularity of the vectors F_p and λ_q in the metric defined by the object G_{pq} is a necessary and sufficient condition for eliminating the need of considering (1.14).

2. We go on to consider the quadratic integrals of a nonholonomic scleronomous mechanical system moving by inertia. The general form of such an integral is

$$b_{ac}\dot{s}^{a}\dot{s}^{c} = C \tag{2.1}$$

In order for (2.1) to be an integral of a nonholonomic mechanical system it is necessary and sufficient [2] to fulfill the following condition:

$$\nabla_a b_{dc} + \nabla_d b_{cd} + \nabla_c b_{ad} = 0 \tag{2.2}$$

Suppose that the quadratic integral has been obtained from the expression $a_{\lambda\mu}\dot{q}^{\lambda}\dot{q}^{\mu}$ after substitution (1.1), then $b_{ac} = a_{\lambda\mu}\alpha_a{}^{\lambda}\alpha_c{}^{\mu}$. By substituting these expressions into condition (2.2), we find

$$\alpha_a{}^{\varkappa}\alpha_d{}^{\lambda}\alpha_c{}^{\mu}(\nabla_{\varkappa}a_{\lambda\mu} + \nabla_{\lambda}a_{\mu\varkappa} + \nabla_{\mu}a_{\varkappa\lambda}) +$$
$$+ a_{\lambda\mu}\alpha_p{}^{\lambda}\,[(B^p_{ad} + B^p_{da})\,\alpha^{\mu}_{c} + (B^p_{dc} + B^p_{ca})\,\alpha^{\mu}_{a} + (B^p_{ca} + B^p_{ac})\,\alpha^{\mu}_{d}] = 0 \tag{2.3}$$

If $a_{\lambda\mu}\dot{q}^{\lambda}\dot{q}^{\mu} = C$ is a quadratic integral of a CFS, we have [9]

$$\nabla_{\varkappa}a_{\lambda\mu} + \nabla_{\lambda}a_{\mu\varkappa} + \nabla_{\mu}a_{\varkappa\lambda} = 0 \tag{2.4}$$

Then, taking (2.4) into account, from (2.3) we obtain

$$a_{\lambda\mu}\alpha^{\lambda}_{p}\,[(B^p_{ad} + B^p_{da})\,\alpha^{\mu}_{c} + (B^p_{dc} + B^p_{cd})\,\alpha^{\mu}_{a} + (B^p_{ca} + B^p_{ac})\,\alpha^{\mu}_{d}] = 0 \tag{2.5}$$

Condition (2.5) is necessary and sufficient for an integral of a CFS to generate a quadratic integral of a nonholonomic system. Just as in the case of linear integrals we see that the condition

$$a_{\lambda\mu}\alpha_p{}^{\lambda}\alpha_a{}^{\mu} = 0 \tag{2.6}$$

is sufficient for a quadratic integral of a CFS to generate a quadratic integral of a nonholonomic system. The general solution of the system of Eqs. (2.6) is

$$a_{\lambda\mu} = \rho_{ab}\omega_{\lambda}{}^{a}\omega_{\mu}{}^{b} + \rho_{pq}\omega_{\lambda}{}^{p}\omega_{\mu}{}^{q} \tag{2.7}$$

where ρ_{ab}, ρ_{pq} are symmetric objects. In the general case formula (2.7) does not yield all the solutions of system (2.5).

Consider Example 2. Condition (1.11) is fulfilled, therefore, condition (2.5) is satisfied identically and consequently each quadratic integral of the CFS generates a quadratic integral of the nonholonomic system. These results are natural if we keep Sumbatov's theorem [10] in mind. It can be immediately established that the constraints are the linear first integrals of a CFS moving by inertia. Consequently, in the notation adopted in [10], we obtain $R_{2j}=0$. Since $Q_{\varkappa} = 0$, we have $R_{0j} = 0$. Then $\varphi = 0$ satisfies the theorem's requirements. Consequently, the Hamilton-Jacobi equations for the CFS and the nonholonomic system coincide. We thus conclude that the phase trajectories of the nonholonomic system are part of the phase trajectories of the CFS. Every integral of the CFS, retaining a constant value on the phase trajectories of this system retains the very same constant value on the trajectories of the nonholonomic system. So that every integral of the CFS generates an integral of the nonholonomic system.

When a mechanical system does not move by inertia, it admits, besides an integral of form (2.1), of a quadratic integral of the form

$$b_{ac}\dot{s}^{a}\dot{s}^{c} + V = C \tag{2.8}$$

where V is a function of the $q^{\varkappa}$. Conditions (2.2) and

$$b_{bc}F^{c} = 0 \tag{2.9}$$

are the conditions for a mechanical system to admit of integrals of form (2.1) [2], while conditions (2.2) and

$$2b_{bc}F^b + \frac{\partial V}{\partial q^\varkappa}\alpha_c{}^\varkappa = 0 \qquad (2.10)$$

for it to admit of integrals of form (2.8). In both cases condition (2.2) goes over into (2.5). Conditions (2.9) and (2.10) impose additional constraints. Keeping (1.15) in mind, for (2.9) we obtain

$$a_{\lambda\mu}\alpha_b\,\alpha_c{}^\mu G^{cd}Q_\varkappa\alpha_d{}^\varkappa = a_{\lambda\mu}\alpha_b{}^\lambda Q_\varkappa(g^{\mu\varkappa} - G^{pq}\alpha_p{}^\mu\alpha_q{}^\varkappa) = a_{\lambda\mu}\alpha_b{}^\lambda Q^\mu = b_{bp}F^p = 0$$

If $a_{\lambda\mu}q^{\cdot\lambda}q^{\cdot\mu} = C$ is an integral of a CFS, then the first term [9] equals zero and (2.9) takes the form

$$b_{bp}F^p = 0 \qquad (2.11)$$

In the same way, for (2.10) we find

$$(2a_{\varkappa\mu}Q^\mu + \partial V\,/\,\partial q^\varkappa)\alpha_c^\varkappa - b_{cp}F^p = 0$$

If $\alpha_{\lambda\mu}q^{\cdot\lambda}q^{\cdot\mu} + V = C$ is an integral of a CFS, then [9] the first term equals zero, so that condition (2.10) turns into (2.11) and, consequently, coincides with (2.9).

Consider Chaplygin's sleigh once more (Example 3). For this system the doubled kinetic energy and the nonholonomic constraint are

$$2T = (x^\cdot + l\cos\varphi\varphi^\cdot)^2 + (y^\cdot + l\sin\varphi\varphi^\cdot)^2 + k^2\varphi^{\cdot 2}, \quad dy = \operatorname{tg}\varphi dx$$

Then the quadratic integral of the CFS is

$$(x^\cdot + l\cos\varphi\varphi^\cdot)^2 + (y^\cdot + l\sin\varphi\varphi^\cdot)^2 = C \qquad (2.12)$$

On the other hand, we can choose α_1 (1, 0,0), α_2 (0, 1, tg φ) as the admissible vectors. If we drop the indices on α_1 and α_2 and use the nonholonomic constraint, we find

$$\omega^1\,(k^2 + l^2,\ l\cos\varphi,\ l\sin\varphi), \qquad \omega^2\,(l\,/\cos\varphi,\ 1,\ \operatorname{tg}\varphi)$$
$$\omega^3\,(0,\ -\operatorname{tg}\varphi,\ 1)$$

We can directly establish that

$$a_{\lambda\mu} = \cos^2\varphi\omega_\lambda{}^2\omega_\mu{}^2 + \cos^2\varphi\omega_\lambda{}^3\omega_\mu{}^3$$

It follows from formula (2.7) that if we substitute $y^\cdot = \operatorname{tg}\varphi\, x^\cdot$ into (2.12), we obtain the nonholonomic system's quadratic integral

$$\cos^{-2}\varphi\,(x^\cdot + l\cos\varphi\varphi^\cdot)^2 = C$$

3. The proposed method can be extended, by using [1], to the first integrals of a nonholonomic system moving by inertia, which have the form

$$b_{ac\ldots d}s^{\cdot a}s^{\cdot c}\ldots s^{\cdot d} = C \qquad (3.1)$$

It can be shown that this is the most general form of a first integral, being a rational entire function of the generalized velocities. Proceeding in exactly the same way as in Sect. 1 and 2, we obtain the necessary and sufficient conditions for a CFS's first integral of the form

$$a_{\lambda\mu\ldots\nu}q^{\cdot\lambda}q^{\cdot\mu}\ldots q^{\cdot\nu} = C \qquad (3.2)$$

to generate a nonholonomic system's first integral. After we have substituted (1.1) into (3.2), we get

$$b_{ac\ldots d} = a_{\lambda\mu\ldots\nu}\alpha_a{}^\lambda\alpha_c{}^\mu\ldots\alpha_d{}^\nu$$

All the arguments in Sect. 1 and 2 extend to the general case. The expressions obtained are cumbersome. The remarks made remain in force when the mechanical system does not move by inertia.

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QUALITATIVE INVESTIGATION OF A DYNAMIC SYSTEM

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We make a qualitative investigation of a dynamic system by bifurcation-theoretic methods [1], using the property of the monotonic rotation of the direction field. We trace the possible bifurcations and the behavior of the bifurcation curves in various sections of the parameter space. The system has been examined before [2, 3], however, a complete qualitative investigation has not been made.

1. Rotation of the field. We examine the system

$$\frac{d\varphi}{dt} = y = P, \qquad \frac{dy}{dt} = \beta - \sin\varphi - 2\alpha s \frac{y}{s^2 + y^2} = Q \tag{1.1}$$

for positive α, β and s. The difference between the direction fields of system (1.1) with parameters β, α_0, s_0 and of an altered system with parameters β, α_1, s_1 for $y \neq 0$ is

$$2\,[s_0 s_1 (\alpha_1 s_0 - \alpha_0 s_1) + (\alpha_1 s_1 - \alpha_0 s_0)\, y^2]\,[(s_0^2 + y^2)(s_1^2 + y^2)]^{-1} \tag{1.2}$$